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TWIST GROUPS OF COMPACT 3-MANIFOLDS

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LET D be a properly imbedded 2-disc in a 3-manifold M. Choosing a neighborhood N of D which is homeomorphic to $D^2 \times I$ with $N \cap \partial M$ corresponding to $\partial D^2 \times I$, we define a *twist about* D to be the homeomorphism t_D such that $t_D(re^{2\pi i\theta}, t) = (re^{2\pi i(\theta+t)}, t)$ if $(re^{2\pi i\theta}, t) \in N$ and $t_D(x) = x$ if $x \notin N$. Note that $t_D|_{\partial M}$ is a Dehn twist about ∂D. There are two directions for twisting, corresponding to the choices of orientation for a regular neighborhood of D. Except for this choice, the isotopy class $\langle t_D \rangle$ in the full mapping class group $\mathcal{M}(M)$ depends only on the ambient isotopy class of D in M. The collection of all isotopy classes of twists generates the *twist group* $\mathcal{T}(M)$, a normal subgroup of $\mathcal{M}(M)$.

It is easy to see that t_D is homotopic to the identity map 1_M and hence $\mathcal{T}(M)$ is contained in the kernel of the natural homomorphism $\mathcal{M}(M) \to \mathrm{Out}(\pi_1(M))$ that takes $\langle h \rangle$ to h_*. But $t_D|_{\partial M}$ is isotopic to $1_{\partial M}$ only when ∂D bounds a disc or a Möbius band in ∂M, hence t_D is often not isotopic to 1_M.

Understanding the twist group is important in studying the mapping class groups of 3-manifolds-with-boundary. In the case of a handlebody (3-ball with 1-handles attached) we have the following theorem of Luft:

THEOREM ([14]). *Let V be an orientable handlebody. Then $\mathcal{T}(V)$ equals the kernel of $\mathcal{M}_1(V) \to Out(\pi_1(V))$, where $\mathcal{M}_1$ indicates orientation-preserving mapping classes.*

This was generalized in [17] to:

THEOREM. *Let M be a compact $\mathbf{P}^2$-irreducible 3-manifold with nonempty boundary, and basepoint $x_0 \in int(M)$. Suppose $h\colon (M, x_0) \to (M, x_0)$ is a homeomorphism with h_* the identity on $\pi_1(M, x_0)$.*

(*a*) *If the local degree of h at x_0 is 1, then h is isotopic (rel x_0) to a product of twist homeomorphisms*

(*b*) *If the local degree of h at x_0 is -1, then M is an I-bundle over a compact 2-manifold. In particular, if M has compressible boundary, then M is a handlebody.*

In the case of non-irreducible 3-manifolds, we have a result which extends a result of Hendriks:

THEOREM ([11]). *Let M be a compact 3-manifold containing no 2-sided projective planes, and let $f\colon (M, \partial M, x_0) \to (M, \partial M, x_0)$ be a map with f_* the identity on $\pi_1(M, x_0)$ and having twisted degree* 1. *Then f is properly homotopic* (rel x_0), *to a homeomorphism $t \circ r \circ h \circ g$, where t is a product of twist homeomorphisms, r is a rotation about an imbedded 2-sphere in M, and h and g are certain homeomorphisms constructed using 2-sphere boundary components of M.*

It is a plausible conjecture that for any compact 3-manifold M, the kernel of $\mathcal{M}(M) \to \mathrm{Out}(\pi_1(M))$ is finitely generated if and only $\mathcal{T}(M)$ is finitely generated. It also seems likely that $\mathcal{M}(M)$ is finitely generated for any compact 3-manifold—this is known for all $\mathbf{P}^2$-irreducible sufficiently large 3-manifolds [10], [17] and for lens spaces [3] and many other 3-manifolds with finite fundamental groups. However, $\pi_1(\mathrm{Homeo}(M))$ is not finitely generated if M is a connected sum of at least three closed aspherical summands [15]. Also, if M_1 is a compact aspherical 3-manifold-with-boundary such that $\mathrm{Out}(\pi_1(M_1))$ is finite and

$\pi_1(M_1)$ admits a surjective homomorphism onto $\mathbf{Z} \times \mathbf{Z}$, then the group of homotopy equivalences $\pi_0(\mathrm{Equiv}(M_1 \# D^3 \# D^3))$ is not finitely generated [16].

In this paper, we shall consider the problem of determining whether $\mathscr{T}(M)$ is finitely generated, showing that finite generation is equivalent to a simple geometric condition involving the boundary of M. Precisely, we say that ∂M is *almost incompressible* if in each boundary component F of M there is at most one simple closed curve (up to isotopy in ∂M) that bounds a disc in M but does not bound a disc or a Möbius band in F. We will prove

COROLLARY 3.3.2. *Let M be a compact 3-manifold. Then $\mathscr{T}(M)$ is finitely generated if and only if ∂M is almost incompressible.*

In view of Luft's theorem quoted above, this shows that $\ker(\mathscr{M}(V) \to \mathrm{Out}(\pi_1(V)))$ is not finitely generated when V is a handlebody of genus ≥ 2, solving Problem 2.4 of Kirby's Stanford problem list [12]. When V is an orientable handlebody of genus 2, this problem was solved by Kramer [13] using methods different from ours. We also determine all the finitely generated twist groups:

THEOREM 4.2. *Let M be a compact connected 3-manifold with finitely generated twist group.*

(a) *If M is a solid Klein Bottle, then $\mathscr{T}(M) \cong \mathbf{Z}/2$*
(b) *If M is $\mathbf{P}^2$-irreducible and not a solid Klein bottle, then $\mathscr{T}(M) \cong \mathbf{Z}^k$ for some $k \geq 0$, and any such group is the twist group of some compact $\mathbf{P}^2$-irreducible 3-manifold*
(c) *If M is irreducible, or more generally if no proper summand of M is a solid Klein bottle, then $\mathscr{T}(M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^l$ for some $k, l \geq 0$, and any such group is the twist group of some compact irreducible 3-manifold*
(d) *$\mathscr{T}(M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^l \times (\mathbf{Z}/4)^m$ for some $k, l, m \geq 0$, and any such group is the twist group of some compact 3-manifold.*

The sections of the paper are organized as follows. In section 1, we use an elementary covering space argument to show that $\mathscr{T}(V)$ is not finitely generated when V is a handlebody of genus ≥ 2. In section 2, we study the case when V is a product-with-handles (defined at the start of section 2). The general case is reduced to the case of products-with-handles in section 3, while the final section treats exclusively the case of $\mathscr{T}(M)$ finitely generated.

We work in the *PL* category without explicit mention. For $A \subseteq X$, the notation $\mathscr{M}(X \operatorname{rel} A)$ means the group of path components of $\{h \in \mathrm{Homeo}(X) \mid h|_A = 1_A\}$.

The letters M and V will always denote connected 3-manifolds, while the letter W will often denote a 3-manifold that is not necessarily connected.

A *disc* in a 3-manifold always means a 2-disc which is either properly imbedded or is imbedded in the boundary.

A *punctured* 3-manifold means a manifold from which the interior of one 3-ball imbedded in the interior of M has been removed. We say N is a *proper summand* of M if $M = N \# N'$ with $N, N' \neq S^3$. The boundary connected sum of M and N is denoted by $M \natural N$; the discs and orientations used to form the boundary connected sum will always be specified, or clear from context.

A *rotation about a 2-sphere* and a *rotation about a projective plane* are homeomorphisms defined similarly to twist homeomorphisms using a 2-sphere or a 2-sided projective plane imbedded in the interior of M, or sometimes imbedded as a boundary component of M. We refer the reader to [9] for facts about these homeomorphisms; here we note only that the square of a rotation about a 2-sphere or projective plane is always isotopic to the identity map, by an isotopy fixed outside a product neighborhood of the 2-sphere or projective plane.

I wish to thank the referee for correcting several mistakes in an earlier version of this paper.

§1. THE TWIST GROUP OF A HANDLEBODY

Throughout this section V will denote a handlebody of genus $g \geq 2$. All homology will have coefficients in the ring R with $R = \mathbf{Z}$ when V is orientable and $R = \mathbf{Z}/2$ when V is nonorientable. It will cause no confusion to simplify notation by suppressing the distinction between a homeomorphism and its isotopy class, and between a simple closed curve and its homology class.

If C is a loop in ∂M, then $t_C: \partial M \to \partial M$ denotes a Dehn twist about C. Recall that for $\alpha \in H_1(\partial M)$, $(t_C)_*(\alpha) = \alpha \pm (\alpha \cdot C)C$ where $\alpha \cdot C \in R$ is the intersection number and the sign depends on the choices of orientation and conventions. In particular, if α and C bound discs in M, then $\alpha \cdot C = 0$ so $(t_C)_*(\alpha) = \alpha$.

THEOREM 1.1. *Let V be a handlebody of genus $g \geq 2$. Then $\mathscr{T}(V)$ is not finitely generated.*

Proof. Let E be a nonseparating properly imbedded 2-disc in V and let $\tilde{V}$ be the infinite cyclic covering whose fundamental group consists of all elements of $\pi_1(V)$ that have intersection number zero with E. We may select nice generators for $H_1(\partial \tilde{V})$, as illustrated in Fig. 1, so that if $s: \tilde{V} \to \tilde{V}$ generates the group of covering transformations, then:

1. $H_1(\tilde{V})$ is a free R-module on $\{s^k a_i | 1 \leq i \leq g-1, k \in \mathbf{Z}\}$
2. $H_1(\partial \tilde{V})$ is a free R-module on $\{c, s^k a_i, s^k b_i | 1 \leq i \leq g-1, k \in \mathbf{Z}\}$
3. kernel $(H_1(\partial \tilde{V}) \to H_1(\tilde{V}))$ is a free R-module on $\{c, s^k b_i | 1 \leq i \leq g-1, k \in \mathbf{Z}\}$

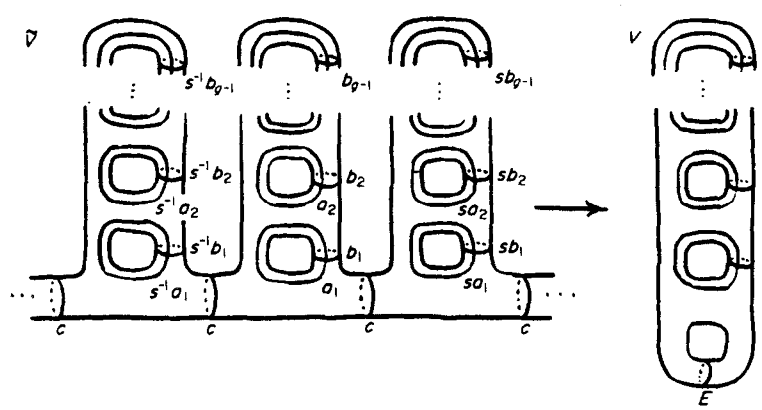

Fig. 1.

For $\langle t \rangle \in \mathscr{T}(V)$, let $\tilde{t}: \tilde{V} \to \tilde{V}$ denote the lift such that $\tilde{t}_*$ is the identity on $H_1(\tilde{V})$. Note that $\widetilde{t_1 t_2} = \tilde{t}_1 \tilde{t}_2$, and that for a twist t_D about $D \subseteq V$, $\tilde{t}_D$ is the lift of t_D that consists of twists about each of the countably many discs that form the preimage of D. Since $\tilde{t}_*$ is the identity, we have

$$(\tilde{t}|_{\partial \tilde{V}})_*(a_1) = a_1 + nc + \sum_{k=-N}^{N} \sum_{i=1}^{g-1} n_{ik} s^k b_i \qquad (*)$$

for some n, $n_{ik} \in \mathbf{Z}$ and some nonnegative integer N. We define the *length* $L(t)$ to be the smallest N for which $(*)$ holds. That is, N is zero if all $n_{ik} = 0$ and otherwise is the largest value of $|k|$ for which some n_{ik} is nonzero. If D is any disc in V, then each of the twists about the preimage discs over D must fix c and the $s^k b_i$ in $H_1(\partial \tilde{V})$, since these elements bound in $\tilde{V}$. It follows that $L(t_1 t_2) \leq \max\{L(t_1), L(t_2)\}$. Consequently, if $L(t)$ were bounded for all t in some generating set of $\mathscr{T}(V)$—in particular, if $\mathscr{T}(V)$ were finitely generated—then $\{L(t) | t \in \mathscr{T}(V)\}$ would be bounded. But let E_N be the disc shown in Fig. 2. It wraps N times around the handle whose cocore is E. One checks that $(\tilde{t}_{E_N})_*(a_1) = a_1 \pm (s^N - 2 + s^{-N})b_1$, hence $L(t_{E_N}) = N$ and the lengths are unbounded. This completes the proof of theorem 1.1. $\square$

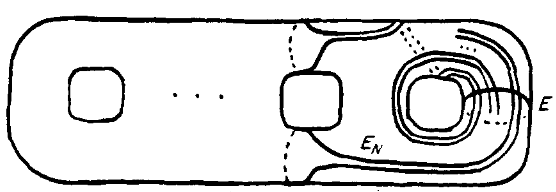

Fig. 2.

We can strengthen theorem 1.1 a bit:

THEOREM 1.2. *Let V be a handlebody of genus $g \geq 2$. Then there is a surjection from $\mathscr{T}(V)$ onto a free abelian group of infinite rank.*

Proof: If V is nonorientable, there is a nonseparating disc E in V so that the infinite cyclic cover formed using V is orientable. Therefore we may always use coefficients $R = \mathbf{Z}$ in the proof of theorem 1.1. With notation as in that proof, we may view $(\tilde{t}|_{\partial\tilde{V}})_*$ as a matrix with respect to the $\mathbf{Z}[s, s^{-1}]$-module generating set $\{b_1, b_2, \dots, b_{g-1}, a_1, a_2, \dots, a_{g-1}, c\}$ of $H_1(\partial\tilde{V})$. Since $(\tilde{t}|_{\partial\tilde{V}})_*$ must preserve the kernel of $H_1(\partial\tilde{V}) \to H_1(\tilde{V})$, this matrix will have the form

$$\left[\begin{array}{c|c|c} I_{g-1} & M & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 & I_{g-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0\dots 0 & n_1 \dots n_{g-1} & 1 \end{array}\right]$$

where I_{g-1} is the identity matrix and M is a $(g-1) \times (g-1)$ matrix with entries in $\mathbf{Z}[s, s^{-1}]$. Observing the effect of multiplying two matrices of this form, we see that the function sending t_D to M is a homomorphism from $\mathscr{T}(V)$ to the *additive* group $M(g-1, \mathbf{Z}[s, s^{-1}])$ of $(g-1) \times (g-1)$ matrices with entries in $\mathbf{Z}[s, s^{-1}]$.

There is a constraint which the entries of M must satisfy. Let $\varepsilon = +1$ or -1 according as V is orientable or not. For $m \in \mathbf{Z}[s, s^{-1}]$, let m^- denote its image under the ring involution defined by sending s to εs^{-1}. We will show that the entries of M satisfy $m_{ji} = m_{ij}^-$.

Suppose $\tilde{D}$ is some lift of D to $\tilde{V}$. Write

$$\partial\tilde{D} = nc + \sum_k n_{1,k} s^k b_1 + \dots + \sum_k n_{g-1,k} s^k b_{g-1} \in H_1(\partial\tilde{V}).$$

In this and the ensuing formulas, the summations range over all values of the indicated letters, and all but finitely many terms are zero. We see that $a_j \cdot s^l \partial\tilde{D} = \varepsilon^l n_{j,-l}$. Since $\tilde{t}_D$ consists of simultaneous twists about the $s^l\tilde{D}$, we have

$$\begin{aligned}(\tilde{t}_D|_{\partial\tilde{V}})_*(a_j) &= a_j + \sum_l (a_j \cdot s^l \partial\tilde{D}) s^l \partial\tilde{D} \\ &= a_j + \sum_l \varepsilon^l n_{j,-l} s^l \partial\tilde{D} \\ &= a_j + \sum_l \varepsilon^l n_{j,l} s^l \partial\tilde{D} \\ &= a_j + \sum_l \varepsilon^l n_{j,l} nc + \sum_{k,l} \varepsilon^l n_{j,l} n_{1,k} s^{k-l} b_1 + \dots\end{aligned}$$

showing that $m_{ij} = \sum_{k,l} \varepsilon^l n_{j,l} n_{i,k} s^{k-l}$. The coefficient of s^q in m_{ij} is $\sum_l \varepsilon^l n_{j,l} n_{i,l+q}$. But the coefficient of s^{-q} in m_{ji} is $\sum_l \varepsilon^l n_{i,l} n_{j,l-q} = \sum_l \varepsilon^{l+q} n_{i,l+q} n_{j,l}$, hence $m_{ji} = m_{ij}^-$.

We have shown that the image of the homomorphism from $\mathscr{T}(V)$ to $M(g-1, \mathbf{Z}[s, s^{-1}])$ lies in

$$A = \{M \in M(g-1, \mathbf{Z}[s, s^{-1}]) \mid m_{ji} = m_{ij}^- \text{ for all } 1 \leq i \leq j \leq g-1\}.$$

It is not hard to observe that A is a countably generated free abelian group. Using various discs in V, one checks that all of these generators are in the image. This proves theorem 1.2. □

§2. THE TWIST GROUP OF A PRODUCT-WITH-HANDLES

A compact connected 3-manifold V is a *product-with-handles* (PWH) if it can be constructed as follows. Take a compact 2-manifold G, no component of which is a 2-sphere, form $G \times I$, and attach 1-handles (at least one) to $G \times \{1\}$. If $F_1, F_2, \ldots, F_n$ are the boundaryless connected components of G, then V has n incompressible boundary components $F_1 \times \{0\}, F_2 \times \{0\}, \ldots, F_n \times \{0\}$, and one compressible boundary component. We usually denote $F_i \times \{0\}$ by F_i and the compressible boundary component by F. Dually, we may regard V as obtained from $F \times I$ by adding 2- and 3-handles which touch $F \times \{0\}$, consequently $\pi_1(F) \to \pi_1(V)$ is surjective. Observe that V is irreducible, but is $\mathbf{P}^2$-irreducible if and only if no component of G is a projective plane. Also, $\pi_1(V) \cong S_1 * S_2 * \ldots * S_n * S$ where $S_i \cong \pi_1(F_i)$ and S is a free group of rank m. The number m is called the *genus* of V. This term is justified by the observation that V is homeomorphic to a boundary connected sum of a genus m handlebody with $F_1 \times I, \ldots, F_n \times I$, where the $F_i \times I$ are all attached to the handlebody along disjoint discs in its boundary. From this observation, the next lemma follows easily.

LEMMA 2.1. *Let V and V' be products-with-handles. Then V is homeomorphic to V' if and only if $\pi_1(V) \cong \pi_1(V')$ and either V and V' are both orientable or V and V' are both nonorientable.*

A PWH V is *of type* (m, n, p) if it has genus m and has n incompressible boundary components of which p are projective planes. We say V is *small* if $(m, n, p) \in \{(1,0,0), (0,2,0), (0,2,1), (0,2,2), (1,1,1), (0,3,3)\}$, otherwise V is *large*. Explicitly, V is type $(1,0,0)$ if V is a solid torus or a solid Klein bottle, of type $(0,2,0)$, $(0,2,1)$, or $(0,2,2)$ if $V = F_1 \times I \,\Box\, F_2 \times I$, of type $(1,1,1)$ when $V = S^1 \times D^2 \,\Box\, \mathbf{P}^2 \times I$, and of type $(0,3,3)$ when V is the unique PWH of genus 0 having three incompressible boundary components which are all projective planes.

The special significance of projective plane boundary components stems from the following observation.

Remark 2.2. If D is a 2-disc with $D \subseteq \mathbf{P}^2 \times \{0\} \subseteq \mathbf{P}^2 \times I$, then a twist homeomorphism about D is isotopic (rel $D \cup \mathbf{P}^2 \times \{1\}$) to a rotation supported in a neighborhood of $\mathbf{P}^2 \times \{1\}$, and hence is isotopic (rel D) to the identity. This may be seen geometrically by making use of an S^1-action on $\mathbf{P}^2$ that rotates D about its center. Consequently, if a properly-imbedded 2-disc $D \subseteq V$ separates V and one component of V split along D is $\mathbf{P}^2 \times I$, then a twist about D is isotopic to the identity on V, and is isotopic (rel $\partial V - F$) to a rotation about a projective plane boundary component of V.

THEOREM 2.3. *If V is a small product-with-handles, then $\mathcal{T}(V)$ is finitely generated. Explicitly,*

(a) *If V is a solid torus then $\mathcal{T}(V) \cong \mathbf{Z}$*
(b) *If V is a solid Klein bottle then $\mathcal{T}(V) \cong \mathbf{Z}/2$*
(c) *If V is of type $(0,2,0)$ then $\mathcal{T}(V) \cong \mathbf{Z}$*
(d) *If V is of type $(0,2,1)$ or $(0,2,2)$ then $\mathcal{T}(V) \cong \{1\}$*
(e) *If V is of type $(0,3,3)$ then $\mathcal{T}(V) \cong \{1\}$*
(f) *If V is of type $(1,1,1)$ then $\mathcal{T}(V) \cong \mathbf{Z}$*

We will break the proof of theorem 2.3 into several lemmas.

LEMMA 2.4. *If V is of type $(0, 2, p)$, so that V is a boundary connected sum of $F_1 \times I$ with $F_2 \times I$ along a disc Δ, then any essential disc in V is ambient isotopic to Δ.*

Proof: Given D, put D transverse to Δ, eliminate simple closed curve intersections by isotopy using irreducibility of V, and eliminate arc intersections using boundary irreducibility of $F_i \times I$. Once D is moved to be disjoint from Δ, it must be parallel to Δ by boundary irreducibility of the $F_i \times I$. □

LEMMA 2.5. *If V is of type* $(0, 3, 3)$, *then any essential disc in V separates V into two components, one of which is homeomorphic to* $\mathbf{P}^2 \times I$.

Proof: This follows easily from theorem 2 of [4]. □

LEMMA 2.6. *If V is of type* $(1, 1, 1)$, *then V contains exactly three isotopy classes of properly imbedded essential 2-discs. One is nonseparating, and the others each separate V into two components, one of which is homeomorphic to* $\mathbf{P}^2 \times I$.

Proof. We begin with a definition. Suppose E_1 and E_2 are disjoint properly imbedded 2-discs in a 3-manifold M, and α is an arc in ∂M intersecting E_1 in one endpoint, E_2 in the other endpoint, with $\operatorname{int}(\alpha)$ disjoint from $E_1 \cup E_2$. We call such an arc *a binding arc* for $E_1 \cup E_2$. If N is a small regular neighborhood of $E_1 \cup \alpha \cup E_2$, then the frontier of N in M consists of three 2-discs. One of these is parallel in N to E_1, another to E_2, and the third is denoted $E(E_1, \alpha, E_2)$ and called the *trough sum* of E_1 and E_2 *along* α. Observe that the isotopy class of $E(E_1, \alpha, E_2)$ depends only on the ambient isotopy class of $E_1 \cup \alpha \cup E_2$, that $E(E_1, \alpha, E_2)$ is isotopic to E_2 if E_1 is parallel to a disc in ∂M, and that $E(E_1, \alpha, E_2)$ is the same as $E(E_2, \alpha, E_1)$.

We regard V as constructed by taking a 3-ball B with three disjoint discs D_1, D_2, and D_3 in ∂B, attaching an orientable 1-handle H along D_1 and D_2, and attaching a $\mathbf{P}^2 \times I$, called P, along D_3. Let H' be the closure of $V - H$ and let P' be the closure of $V - P$. Let $c \subseteq \partial H' - \operatorname{int}(D_1 \cup D_2)$ be a binding arc for $D_1 \cup D_2$ that runs over P so that $D_1 \cup c \cup D_2$ is not isotopic into P'. Define $E_1 = D_1$, $E_2 = E(D_1, c, D_2)$, and $E_3 = D_3$. These are shown in Fig. 3. We will prove that any essential disc Δ in V is isotopic to one of E_1, E_2, or E_3. This will prove the lemma, since E_1 is nonseparating, E_2 separates V into a $\mathbf{P}^2 \times I$ and a solid Klein bottle, and E_3 separates V into a $\mathbf{P}^2 \times I$ and a solid torus.

Given Δ, we may assume Δ intersects $D_1 \cup D_2 \cup D_3$ transversely in arcs. An arc of intersection that is outermost on $D_1 \cup D_2 \cup D_3$ gives a decomposition of Δ as $E(\Delta_1, \alpha, \Delta_2)$ where Δ_1 and Δ_2 have fewer total arcs of intersection with $D_1 \cup D_2 \cup D_3$ than Δ had (i.e. if the arc cuts off a disc $D \subseteq D_i$ with interior disjoint from Δ, then the frontier of a regular neighborhood of $\Delta \cup D$ consists of Δ, Δ_1, and Δ_2). Iterating, we obtain a way of constructing Δ as an iterated trough sum, starting from discs which are disjoint from $D_1 \cup D_2 \cup D_3$ and hence either inessential or isotopic to E_1 or E_3. Therefore, it suffices to show that a trough sum of two discs, each of which is isotopic to a disc in $\mathscr{E} = \{E_1, E_2, E_3, \text{inessential 2-disc}\}$, is isotopic to a disc in $\mathscr{E}$.

Clearly E (inessential 2-disc, α, inessential 2-disc) is always inessential. If E_2' and E_2'' are isotopic to E_2, then $E(E_2', \alpha, E_2'')$ is always inessential, and similarly for $E(E_3', \alpha, E_3'')$. For $1 \leq i \leq 3$, E(inessential 2-disc, α, E_i) is isotopic to E_i. Suppose now that E_1' and E_1'' are

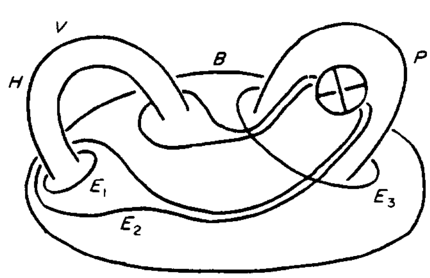

Fig. 3.

isotopic to E_1, then $E(E_1', \alpha', E_1'')$ is isotopic to $E(D_1, \alpha, D_2)$ for some α. If α runs over H, then $E(D_1, \alpha, D_2)$ is inessential. If α lies in $\partial H' - \mathrm{int}(D_1 \cup D_2)$, then either it is isotopic in $\partial H' - \mathrm{int}(D_1 \cup D_2)$ to an arc missing P, in which case $E(D_1, \alpha, D_2)$ is isotopic to E_3, or it is isotopic to the arc we called c, in which case $E(D_1, \alpha, D_2)$ is isotopic to E_2. Next, we consider $E(E_1, \alpha, E_2)$. The boundary of E_2 separates the compressible boundary component of V into a punctured Klein bottle and a Möbius band, with ∂E_1 an orientation-preserving essential loop in the Klein bottle. There are only two isotopy classes (keeping endpoints in $\partial E_1 \cup \partial E_2$) of binding arcs α, and for each of these, $E(E_1, \alpha, E_2)$ is isotopic to E_1. Similarly, $E(E_1, \alpha, E_3)$ is isotopic to E_1. Finally, ∂E_2 and ∂E_3 are not isotopic to disjoint loops (this can easily be proved by examining their lifts to certain covering spaces of ∂V) so the trough sum of E_2 and E_3 cannot occur. This completes the proof of lemma 2.6. □

We can now deduce theorem 2.3 without difficulty. Cases (a) and (b) are easy since any essential disc in a solid torus or solid Klein bottle is isotopic to a meridinal disc. Cases (c) and (d) follow from remark 2.2 and lemma 2.4—when there are no projective plane boundary components, the mapping class of a twist about the essential disc has infinite order when restricted to ∂V and hence has infinite order in $\mathcal{T}(V)$. Case (e) follows from lemma 2.5 and remark 2.2, and case (f) from lemma 2.6 and remark 2.2. This completes the proof of theorem 2.3. □

THEOREM 2.7. *If V is a large product-with-handles, then $\mathcal{T}(V)$ is not finitely generated.*

Proof: Let V be a large PWH of type (m, n, p), with incompressible boundary components $F_1, F_2, \ldots, F_n$. The proof of theorem 2.7 is very similar to the proof of theorem 1.1, although the covering space $\tilde{V}$ must be selected with care. We will describe the covering space $\tilde{V}$ and the disc E_N for each of the cases, and omit the remaining minor modifications to the argument.

Case I. $m \geq 2$
Take E to be the cocore of one of the non-separating 1-handles and $\tilde{V}$ to be the infinite cyclic cover constructed from E as in theorem 1.1. The disc E_N is also quite similar.

Case II. $m = 1$ and $n = 1$ (hence $F_1 \neq \mathbf{P}^2$ since V is large)
Choose a simple closed curve C in $F_1 \times \{1\} \cap \partial V$ such that $E = C \times I \subseteq F_1 \times I$ is a properly-imbedded nonseparating annulus in V. Form the infinite cyclic cover using E, and for E_N use the disc shown in Fig. 4.

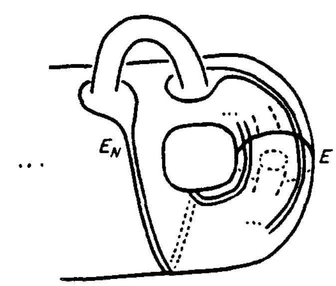

Fig. 4.

Case III. $m = 1$ and $n \geq 2$

Form a double covering V_1 by opening up $F_1 \times I$ and $F_2 \times I$ into double covers. In V_1 let E be one of the discs in the preimage of the cocore of the 1-handle of V. Form an infinite cyclic covering $\tilde{V}$ of V_1 using E. ($\tilde{V}$ is irregular covering of V.) Let a_1 and E_N be as shown in Fig. 5.

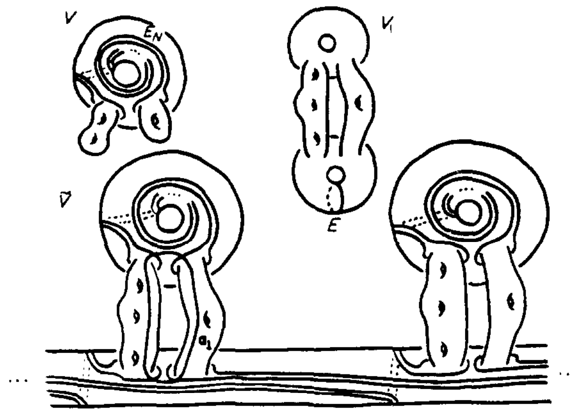

Fig. 5.

Case IV. $m = 0$ (hence $n \geq 3$) and some F_i, say F_3, is not a projective plane. Proceed as in Case III, using $F_3 \times I$ in the role of the 1-handle and a nonseparating annulus in the role of E.

Case V. $m = 0$ and all F_i are projective planes (hence $n = p \geq 4$). Form V_1 as in Case III by opening up $F_1 \times I$ and $F_2 \times I$. The covering $\tilde{V}$ is a regular covering of V_1 having covering transformations $\mathbf{Z}/2 * \mathbf{Z}/2$, obtained by opening up one lift of $F_3 \times I$ and one lift of $F_4 \times I$ in V_1. The loop a_1 and the discs E_N are similar to those in Case III. □

From lemmas 2.4, 2.5, and 2.6, it is clear that if V is small PWH, then ∂V is almost incompressible. On the other hand, the discs called E_N in the proof of theorem 2.7 show that if V is a large PWH, then ∂V is not almost incompressible. Therefore we may combine theorems 2.3 and 2.7 into a succinct statement:

THEOREM 2.8. *Let V be a product-with-handles. Then $\mathcal{T}(V)$ is finitely generated if and only if ∂V is almost incompressible.*

Let $\mathcal{T}(V \operatorname{rel} \partial V - F)$ be the subgroup of $\mathcal{M}(V \operatorname{rel} \partial V - F)$ generated by twist homeomorphisms. We need the following technical results for later use.

LEMMA 2.9. *Let M be a compact 3-manifold and let G be a union of aspherical boundary components of M. Assume that $\pi_1(M)$ is centerless and that for each component G_i of G, the kernel of $\pi_1(G_i) \to \pi_1(M)$ contains no central element of $\pi_1(G_i)$. Let t be a product of twists and rotations about 2-spheres and projective planes, with $t|_G = 1_G$ and t isotopic to 1_M. Then t is isotopic to 1_M (rel G).*

Proof: Choose basepoints $x_0 \in \operatorname{int}(M)$ and $g_i \in G_i$, and arcs α_i running from x_0 to g_i. We may assume x_0 is fixed by t, and since $\pi_1(M)$ is centerless, we may choose an isotopy H from t to 1_M that is (rel x_0). Since t is a product of twists and rotations, we have $t(\alpha_i)$ homotopic to α_i(rel $x_0 \cup g_i$). But $t(\bar{\alpha}_i) * \alpha_i$ is the image of the trace of $H|_{G_i}$ at g_i. Since this trace is a central element of $\pi_1(G_i)$, our hypothesis shows it is trivial in $\pi_1(G_i)$. By results of Hamstrom [5], [6], [7], taking the trace gives an isomorphism π_1 (Homeo $(G_i)) \to \operatorname{Center}(\pi_1(G_i))$ so we can deform H to an isotopy (rel G). □

PROPOSITION 2.10. *Let V be a product-with-handles. Then*

(a) *There is an exact sequence*

$$1 \to (\mathbf{Z}/2)^k \to \mathcal{T}(V \operatorname{rel} \partial V - F) \to \mathcal{T}(V) \to 1$$

in which the kernel is the central subgroup generated by rotations about projective plane boundary components.

(b) *The restriction $\mathcal{T}(V) \to \mathcal{M}(F)$ is injective.*

(c) *There is an exact sequence*

$$1 \to (\mathbf{Z}/2)^k \to \mathscr{T}(V\ rel\ \partial V - F) \to \mathscr{M}(F)$$

in which the kernel is the central subgroup generated by rotations about projective plane boundary components.

Proof. (a) Since any disc with boundary in $\partial V - F$ is inessential, any element of $\mathscr{T}(V)$ is isotopic to a product of twists about discs with boundary in F. Therefore $\mathscr{T}(V \text{ rel } \partial V - F) \to \mathscr{T}(V)$ is surjective. Suppose $\langle t \rangle \in \mathscr{T}(V \text{ rel } \partial V - F)$ and H is an isotopy from t to 1_V. By lemma 2.9, we may assume H is (rel G) where G is the union of the components of $\partial V - F$ that are not projective planes. Let R be the homeomorphism of V that is a rotation about each projective plane boundary component P for which $H|_{P \times I}$ represents the nontrivial element of $\pi_1(\text{Homeo}(P), 1_P) \cong \mathbf{Z}/2$ [6]. Then Rt is isotopic to $1_V (\text{rel } \partial V - F)$ so $\langle t \rangle = \langle R \rangle^{-1} = \langle R \rangle$ in $\mathscr{T}(V \text{ rel } \partial V - F)$. Since rotations about the projective plane boundary components commute and have order 2 in $\mathscr{T}(V \text{ rel } \partial V - F)$, the result follows.

(b) Let $\langle h \rangle \in \mathscr{T}(V)$ with $h|_F = 1_F$. If D is any properly-imbedded disc in V with $\partial D \subseteq F$, then $h|_{\partial D} = 1_{\partial D}$ implies, using irreducibility of V, that h is isotopic to a homeomorphism that fixes $F \cup D$. Thus we may assume h fixes a collection of discs that cuts V into 3-balls and $F_i \times I$'s. On the 3-balls we use the Alexander isotopy to deform h to the identity. On the $F_i \times I$'s, h is the identity on the boundary, so is isotopic, keeping fixed the end that meets the collection of discs, to the identity. For aspherical F_i, this follows from [18] and [8]. For $F_i = \mathbf{P}^2$, note that if A is the annulus that represents the generator of $H_2(\mathbf{P}^2 \times I, \mathbf{P}^2 \times \partial I; \mathbf{Z}/2)$ and h is moved slightly, then by duality $h(A) \cap A$ contains an arc that runs from $\mathbf{P}^2 \times \{0\}$ to $\mathbf{P}^2 \times \{1\}$. This shows that if $*$ is the basepoint of $\mathbf{P}^2$, then h is isotopic (rel $\mathbf{P}^2 \times \partial \mathbf{I}$) to a map fixing $* \times I$. From here it is not difficult to deform h to the identity keeping one end fixed. Putting together these isotopies on the pieces of V gives an isotopy from h to 1_V.

(c) is immediate from (a) and (b). □

PROPOSITION 2.11. *Let V be a punctured handlebody. Then*

(a) *there is an exact sequence*

$$1 \to (\mathbf{Z}/2)^k \to \mathscr{T}(V \text{ rel } \partial V - F) \to \mathscr{T}(V) \to 1$$

in which the kernel is generated by a rotation about the 2-sphere boundary component S and $k = 0$ or 1

(b) *the restriction $\mathscr{T}(V) \to \mathscr{M}(F)$ is injective*

(c) *there is an exact sequence*

$$1 \to (\mathbf{Z}/2)^k \to \mathscr{T}(V\ rel\ \partial V - F) \to \mathscr{M}(F)$$

in which the kernel is generated by a rotation about the 2-sphere boundary component S and $k = 0$ or 1.

Proof. (a) Similar to proposition 2.10 (a); $\pi_1(M)$ might not be centerless, but lemma 2.9 is not needed.

(b) Let t be a product of twists on V with $t|_F$ isotopic to 1_F. We may assume $t|_S = 1_S$. Choose basepoints $s_0 \in S$ and $x_0 \in F$ fixed by all the twists in t. Let W be the handlebody obtained by filling in S with a 3-ball E and let $\hat{t}\colon W \to W$ be the homeomorphism obtained by extending t using the identity on E. By proposition 2.10(b), $\hat{t}$ is isotopic to 1_W. If W is not a solid torus or solid Klein bottle, then $\pi_1(F)$ is centerless so we may assume the isotopy from $\hat{t}$ to 1_W is (rel x_0). If W is a solid torus or solid Klein bottle, then for each central element of $\pi_1(F, x_0)$ there is a circular isotopy of W with trace at x_0 equal to this element. Therefore, in this case we may still assume $\hat{t}$ is isotopic to 1_W(rel x_0). For this isotopy, the trace at s_0 is also trivial. This allows us to find an isotopy from $\hat{t}$ to 1_W that leaves E invariant, and therefore restricts on V to an isotopy from t to 1_V.

(c) is immediate from (a) and (b). □

LEMMA 2.12. *Let K^* be a punctured solid Klein bottle with $\partial K^* = K \cup S$. Then $\mathcal{T}(K^* \text{ rel } S) \cong \mathbf{Z}/4$. It is generated by a twist about a meridinal disc D, and $\langle t_D^2 \rangle = \langle r_S \rangle$ in $\mathcal{T}(K^* \text{ rel } S)$.*

Proof. Figure 6 illustrates an isotopy from t_D^2 to r_S (rel S). In Fig. 6(a), t_D^2 is represented by twists about two meridinal discs D_1 and D_2. In Fig. 6(b), D_2 has been isotoped around K^*, returning on the other side of D_1 but reversed with respect to a local orientation near D_1. In Fig. 6(c), the twist about this disc has been isotoped to the product of r_S and a twist that is isotopic to the inverse of t_{D_1}. From proposition 2.11(a), the kernel of $\mathcal{T}(K^* \text{ rel } S) \to \mathcal{T}(K^*) \cong \mathbf{Z}/2$ is generated by $\langle r_S \rangle$, so it remains only to show $\langle r_S \rangle \neq \langle 1_{K^*} \rangle$ in $\mathcal{T}(K^* \text{ rel } S)$. By an argument similar to that of lemma 2.9, if r_S is isotopic to 1_{K^*} (rel S), then it is also isotopic to 1_{K^*} (rel ∂K^*). But by section IV.4.3 of [9], r_S is not even homotopic to 1_{K^*} (rel ∂K^*). □

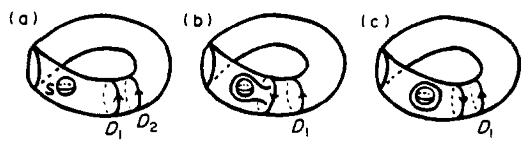

Fig. 6.

§3. TWIST GROUPS OF 3-MANIFOLDS

3.1. *The rotation subgroup*

LEMMA 3.1.1. *Let D be a disc in M and let F be a closed incompressible surface in the interior of M. Then there is a disc D' in M, disjoint from F with $\langle t_D \rangle = \langle r \rangle \langle t_{D'} \rangle$ in $\mathcal{M}(M)$, where r is a product of rotations about 2-spheres disjoint from D' and from F.*

Proof. We may assume D intersects F transversely. Since F is incompressible, either $D \cap F$ is empty or there is a circle $C \subseteq D \cap F$ bounding a disc E in F with int(E) disjoint from D. We can use E to find a 2-sphere Σ and a disc D_1 so that $\langle t_D \rangle = \langle r_\Sigma \rangle \langle t_{D_1} \rangle$ and $\Sigma \cup D_1$ intersects F in fewer circles than D does. Explicitly, let N be a small regular neighborhood of $D \cup E$, then the frontier of N in M consists of Σ, D_1, and a disc parallel to D. A similar process can be used to write $\langle r_\Sigma \rangle = \langle r_{\Sigma'} \rangle \langle r_{\Sigma''} \rangle$ where $\Sigma' \cup \Sigma''$ has fewer circles of intersection with F than Σ has. Repeating, we arrive at a disc D', disjoint from F, and 2-spheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$, disjoint from F and D', so that $\langle t_D \rangle = \langle r_{\Sigma_1} \rangle \langle r_{\Sigma_2} \rangle, \ldots, \langle r_{\Sigma_m} \rangle \langle t_{D'} \rangle$. □
A similar argument shows

LEMMA 3.1.2. *Let r_Σ be a rotation about the 2-sphere $\Sigma \subseteq M$ and let F be a closed incompressible surface in the interior of M. Then there is a product r of rotations about 2-spheres disjoint from F so that $\langle r_\Sigma \rangle = \langle r \rangle$ in $\mathcal{M}(M)$.*

Let $\mathcal{R}(M)$ be the subgroup of $\mathcal{M}(M)$ generated by rotations about imbedded 2-spheres in M.

PROPOSITION 3.1.3. *Let M be a compact 3-manifold. Then*

(a) *$\mathcal{R}(M)$ is a normal subgroup of $\mathcal{M}(M)$*
(b) *$\mathcal{R}(M) \cong (\mathbf{Z}/2)^k$ for some nonnegative integer k*
(c) *If $\partial M \neq \phi$ then $\mathcal{R}(M)$ is a central subgroup of $\mathcal{T}(M)$.*

Proof. (a) If $\langle h \rangle \in \mathcal{M}(M)$ and r_S is a rotation about the imbedded 2-sphere S, then $\langle h \rangle \langle r_S \rangle \langle h^{-1} \rangle = \langle r_{h(S)} \rangle$
(b) Let $\mathcal{S}$ be a union of finitely-many disjoint imbedded 2-spheres such that for each component W of M split along $\mathcal{S}$, the manifold $\hat{W}$ obtained by filling in the 2-sphere

boundary components of W with 3-balls is irreducible. This can be achieved by taking 2-spheres that factor M into prime summands and adding a 2-sphere fiber from each prime summand that is a 2-sphere bundle over S^1. Now if S is any 2-sphere in M that is disjoint from $\mathscr{S}$, then there is a submanifold $M_0 \subseteq M$ which is a 3-ball with some open 3-balls removed such that $\partial M_0 \subseteq \mathscr{S} \cup S$. Therefore r_S is isotopic to some product of rotations about spheres in $\mathscr{S}$. Each rotation about a sphere in $\mathscr{S}$ has order two in $\mathscr{M}(M)$, and they commute since the spheres in $\mathscr{S}$ are disjoint. But by lemma 3.1.2, any rotation is isotopic to a product of rotations about 2-spheres disjoint from $\mathscr{S}$, so (b) follows.

(c) Let Σ be an imbedded 2-sphere and choose an imbedded arc α running from Σ to ∂M and intersecting them only in its endpoints. Let N be a regular neighborhood of $\Sigma \cup \alpha$, then the frontier of N in M consists of a disc D and a 2-sphere parallel to Σ. A twist about D is isotopic to a rotation about Σ, so $\langle r_\Sigma \rangle = \langle t_D \rangle \in \mathscr{T}(M)$. Thus $\mathscr{R}(M) \subseteq \mathscr{T}(M)$. Suppose $\langle t_E \rangle \in \mathscr{T}(M)$ and r_Σ is any rotation. Applying lemma 3.1.1 with $F = \Sigma$ gives a disc D' disjoint from Σ and a product r of rotations about 2-spheres disjoint from Σ so that $\langle t_D \rangle = \langle r \rangle \langle t_{D'} \rangle$, which obviously commutes with $\langle r_\Sigma \rangle$. Therefore $\mathscr{R}(M)$ is central in $\mathscr{T}(M)$. □

3.2. *Incompressible neighborhoods*

PROPOSITION 3.2.1. *Let F be a compressible boundary component of an irreducible 3-manifold M. Then there is a connected codimension zero submanifold V of M such that*

(1) $V \cap \partial M = F$

(2) *$\partial V - F$ is contained in the interior of M and is incompressible in M*

(3) *V is a product-with-handles*

Proof. This differs from the existence parts of theorem 2.1 of [2] and theorem 1.1.1 of [17] only in that we are allowing 2-sided projective planes, but the arguments can be extended to this case with only minor changes. □

Now let M be any compact 3-manifold, factored into prime summands. Each compressible boundary component F lies in some irreducible summand M_1 of M. If M_1 is not a handlebody, let V_F be a product-with-handles in M_1 satisfying the conclusion of proposition 3.2.1. Since M_1 is not a handlebody, $\partial V_F - F$ is nonempty and therefore V_F may be chosen to be disjoint from the 3-balls removed from M_1 to form M. That is, we may assume V_F lies in M. If M_1 is a handlebody, let $V_F = M_1 = M$ if $M = M_1$, otherwise let V_F be a neighborhood of F in M which is a punctured handlebody. Finally, if F is incompressible, let V_F be a collar neighborhood of F in M. We may select the V_F's sequentially to insure they are disjoint. Let W be the union of the V_F's for all boundary components of M. A neighborhood W of ∂M constructed in this way is called an *incompressible neighborhood* of ∂M. Note that $\partial W - \partial M$ is incompressible in M, and hence a loop in ∂M bounds a disc in M if and only if it bounds a disc in W. Consequently, ∂M is almost incompressible in M if and only if ∂W is almost incompressible in W.

3.3. *The main results*

Let M be a compact 3-manifold and let W be an incompressible neighborhood of ∂M. Then extending homeomorphisms using the identity on $M - W$ induces a homomorphism $\mathscr{T}(W \text{ rel } \partial W - \partial M) \to \mathscr{T}(M)$. Let Φ be the composite $\mathscr{T}(W \text{ rel } \partial W - \partial M) \to \mathscr{T}(M) \to \mathscr{T}(M)/\mathscr{R}(M)$.

THEOREM 3.3.1. *Φ: $\mathscr{T}(W \text{ rel } \partial W - \partial M) \to \mathscr{T}(M)/\mathscr{R}(M)$ is surjective. The kernel is contained in the subgroup generated by rotations about projective planes in the boundary of product-with-handles components of W and by rotations about 2-spheres in the boundary of punctured handlebody components of W, hence is a finite group isomorphic to $(\mathbf{Z}/2)^k$ for some k.*

Proof. Surjectivity follows by taking $F = \partial W - \partial M$ in lemma 3.1.1. If $\langle t \rangle$ is in the kernel, then $t|_{\partial M}$ is isotopic to the identity. Applying propositions 2.10(c) and 2.11(c) gives the result. □

COROLLARY 3.3.2. *$\mathcal{T}(M)$ is finitely generated if and only if ∂M is almost incompressible.*

Proof. We have observed that ∂M is almost incompressible if and only if ∂W is almost incompressible in W. By theorem 2.8 and using propositions 2.10 and 2.11, this is the case if and only if $\mathcal{T}$ (W rel $\partial W - \partial M$) is finitely-generated. The corollary now follows from theorem 3.3.1 and proposition 3.1.3(b). □

COROLLARY 3.3.3. *Suppose that M contains no 2-sided projective planes, and let W be an incompressible neighborhood of ∂M. Then there is an exact sequence*

$$1 \to \mathcal{R}(M) \to \mathcal{T}(M) \to \mathcal{T}(W) \to 1.$$

In particular, if M is $\mathbf{P}^2$-irreducible then $\mathcal{T}(M) \cong \mathcal{T}(W)$.

Proof. By theorem 3.3.1 the kernel of $\mathcal{T}(W \text{ rel } \partial W - \partial M) \to \mathcal{T}(M)/\mathcal{R}(M)$ is in the rotation subgroup, so there is an induced surjection $\mathcal{T}(W) \to \mathcal{T}(M)/\mathcal{R}(M)$. But by propositions 2.10(c) and 2.11(c), the composite $\mathcal{T}(W) \to \mathcal{T}(M)/\mathcal{R}(M) \to \mathcal{M}(\partial M)$ is injective, hence the induced surjection is an isomorphism. □

We note that in corollary 3.3.3 it is sufficient to assume only that W has no projective plane boundary components, except possibly in components of W that are collar neighborhoods of projective plane boundary components of M. Also, if in corollary 3.3.3 M has no proper summands that are handlebodies, then propositions 2.10(a) and 2.11(a) show that $\mathcal{T}(W \text{ rel } \partial W - \partial M) \to \mathcal{T}(W)$ is an isomorphism. Then, $\mathcal{T}(W \text{ rel } \partial W - \partial M) \to \mathcal{T}(M)$ induces a splitting of the exact sequence, giving $\mathcal{T}(M) \cong \mathcal{R}(M) \times \mathcal{T}(W)$.

COROLLARY 3.3.4. *Let M be a compact 3-manifold. Then $\mathcal{T}(M)$ is the normal closure of finitely many elements of $\mathcal{M}(M)$.*

Proof. When V is a punctured handlebody or a product-with-handles with compressible boundary component F, it is not hard to see that under the action of $\mathcal{M}(V \text{ rel } \partial V - F)$, there are only finitely many isotopy classes of discs in V. Since $f t_D f^{-1} = t_{f(D)}$, this implies that $\mathcal{T}(W \text{ rel } \partial W - \partial M)$ is the normal closure in $\mathcal{M}(W \text{ rel } \partial W - \partial M)$ of finitely many twists. Theorem 3.3.1 shows that the image of $\mathcal{T}(W \text{ rel } \partial W - \partial M)$ together with $\mathcal{R}(M)$ generate $\mathcal{T}(M)$. Because all elements of $\mathcal{M}(W \text{ rel } \partial W - \partial M)$ extend to M, the result follows. □

§4. FINITELY GENERATED TWIST GROUPS

LEMMA 4.1. *Let W be an incompressible neighborhood of ∂M and suppose $\mathcal{T}(W)$ is finitely generated. Then $\mathcal{T}(W \text{ rel } \partial W - \partial M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^l \times (\mathbf{Z}/4)^m$ for some $k, l, m \geq 0$, where m is the number of components of W that are punctured solid Klein bottles, $l = 0$ unless W is a solid Klein bottle or has projective plane boundary components.*

Proof. The components of W that are collar neighborhoods of boundary components of M contribute nothing to $\mathcal{T}(W \text{ rel } \partial W - \partial M)$. The components that are small products-with-handles contribute direct factors of the form $\mathbf{Z}^{k_1} \times (\mathbf{Z}/2)^{l_1}$, according to theorem 2.3 and proposition 2.10(a). If V is a punctured handlebody component of W, then it must be a punctured solid Klein bottle or punctured solid torus. In the former case, lemma 2.12 shows that V contributes a $\mathbf{Z}/4$ factor; in the latter case, r_S is isotopic to $1_V(\text{rel } S)$ so proposition 2.11(a) and theorem 2.3 show $\mathcal{T}(V \text{ rel } \partial V - S) \cong \mathbf{Z}$. This establishes the assertion about m. The assertion about l follows using theorem 2.3 and proposition 2.10(a). □

THEOREM 4.2. *Let M be a compact connected 3-manifold with finitely generated twist group.*

(a) *If M is a solid Klein bottle, then $\mathcal{T}(M) \cong \mathbf{Z}/2$*
(b) *If M is $\mathbf{P}^2$-irreducible and not a solid Klein bottle, then $\mathcal{T}(M) \cong \mathbf{Z}^k$ for some $k \geq 0$, and any such group is the twist group of some compact $\mathbf{P}^2$-irreducible 3-manifold*
(c) *If M is irreducible, or more generally if no proper summand of M is a solid Klein bottle, then $\mathcal{T}(M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^l$ for some $k, l \geq 0$, and any such group is the twist group of some compact irreducible 3-manifold*
(d) *$\mathcal{T}(M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^l \times (\mathbf{Z}/4)^m$ for some $k, l, m \geq 0$, and any such group is the twist group of some compact 3-manifold.*

PROOF. (a) This is theorem 2.3(b).
(b) From corollary 3.3.3, $\mathcal{T}(M) \cong \mathcal{T}(W)$, and by theorem 2.3, $\mathcal{T}(W) \cong \mathbf{Z}^k$. For realization, let F_i be closed connected aspherical 2-manifolds for $0 \leq i \leq k$, and form M from $\cup_{i=0}^{k} F_i \times I$ by identifying a disc in $F_i \times \{1\}$ with a disc in $F_{i+1} \times \{0\}$ for $0 \leq i \leq k-1$. For this M, W is homeomorphic to a disjoint union $F_0 \times I \cup (F_0 \times I \square F_1 \times I) \cup (F_1 \times I \square F_2 \times I) \cup \ldots \cup (F_{k-1} \times I \square F_k \times I) \cup F_k \times I$, so theorem 2.3 shows $\mathcal{T}(W) \cong \mathbf{Z}^k$.
(c) and (d) Consider the homomorphism $\mathcal{T}(W \text{ rel } \partial W - \partial M) \to \mathcal{T}(M)$ discussed in section 3.3. Since $\mathcal{R}(M)$ is central in $\mathcal{T}(M)$, there is a homomorphism $\mathcal{R}(M) \times \mathcal{T}(W \text{ rel } \partial W - \partial M) \to \mathcal{T}(M)$. By theorem 3.3.1, this homomorphism is surjective and has finite kernel. Therefore proposition 3.1.3(b) and lemma 4.1 show that $\mathcal{T}(M)$ has the asserted form. It remains to realize these groups as twist groups for the appropriate kinds of 3-manifolds.

For (c), let $F_{k+1}, F_{k+3}, \ldots, F_{k+2l-1}$ be projective planes, let $F_0, F_1, \ldots, F_k, F_{k+2}, F_{k+4}, \ldots, F_{k+2l}$ be closed connected aspherical 2-manifolds, and construct M as in (b) above. There is an incompressible neighborhood W of ∂M having components $W_0, W_1, \ldots, W_{k+2l+1}$ where $W_0 \approx F_0 \times I$, $W_{k+2l+1} \approx F_{k+2l} \times I$, and $W_i \approx F_{i-1} \times I \square F_i \times I$ for $1 \leq i \leq k+2l$. Thus for $1 \leq i \leq k$, W_i is of type $(0,2,0)$ so $\mathcal{T}(W_i \text{ rel } \partial W_i - \partial M) \cong \mathbf{Z}$ using theorem 2.3 and proposition 2.10(a), while for $k+1 \leq i \leq k+2l$, W_i is of type $(0,2,1)$ so $\mathcal{T}(W_i \text{ rel } \partial W_i - \partial M) \cong (\mathbf{Z}/2)^{l_i}$ for $l_i = 0$ or 1 (in fact, we will see that $l_i = 1$) again by theorem 2.3 and proposition 2.10(a). Since M is irreducible, theorem 3.3.1 gives a surjection $\mathcal{T}(W \text{ rel } \partial W - \partial M) \to \mathcal{T}(M)$ with finite kernel. Since a rotation about the projective plane in W_{k+2j-1} is isotopic in M to a rotation about the projective plane in W_{k+2j} for $1 \leq j \leq l$, this shows $\mathcal{T}(M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^{l_0}$ for some $l_0 \leq l$. On the other hand, the projective planes $F_{k+1} \times \{\frac{1}{2}\}, F_{k+3} \times \{\frac{1}{2}\}, \ldots, F_{k+2l-1} \times \{\frac{1}{2}\}$ form a projectively minimal Epstein system for M [9, section II.3]. By V.4.4 of [9], the rotations about these projective planes generate a subgroup isomorphic to $(\mathbf{Z}/2)^l$ in π_0 (Equivalences $(M \text{ rel } \partial M)$), and hence generate a subgroup isomorphic to $(\mathbf{Z}/2)^l$ in $\mathcal{M}(M \text{ rel } \partial M)$. Lemma 2.9 now implies they still generate a subgroup isomorphic to $(\mathbf{Z}/2)^l$ in $\mathcal{M}(M)$, hence $l_0 \geq l$ and therefore $\mathcal{T}(M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^l$.
For (d), form the connected sum of the M from part (c) with m solid Klein bottles. A similar analysis and use of results from [9] proves that $\mathcal{T}(M) \cong \mathbf{Z}^k \times (\mathbf{Z}/2)^l \times (\mathbf{Z}/4)^m$. □
Theorem 4.2 has the following curious consequence.

COROLLARY 4.3. *Let M be a compact 3-manifold. Then $\mathcal{T}(M)$ is finitely generated if and only if it is abelian.*

Proof. Theorem 4.2 implies the only if direction. Conversely, if $\mathcal{T}(M)$ were infinitely generated and abelian, then theorem 3.3.1 and proposition 2.10 or 2.11 would show that $\mathcal{M}(\partial M)$ contains an infinitely generated abelian subgroup, which is impossible by [1]. Alternatively, if $\mathcal{T}(M)$ is infinitely generated then W must have a component which is either a large product-with-handles or a possibly punctured handlebody of genus $g \geq 2$, and in either case one can easily find two twists which don't commute up to isotopy. □

REFERENCES

1. J. Birman, A. Lubotzky and J. McCarthy: Abelian and solvable subgroups of the mapping class group. *Duke Math. J.* **50** (1983), 1107–1120.
2. F. Bonahon: Corbordism of automorphisms of surfaces. *Ann. Sci. Ecole Norm. Sup.* (4) **16** (1983), 237–270.
3. F. Bonahon: Diffeotopies des espaces lenticulaires, *Topology* **22** (1983), 305–314.
4. J. Gross: The decomposition of 3-manifolds with several boundary components. *Trans. Am. Math. Soc.* **147** (1970), 561–572.
5. M.-E. Hamstrom: The space of homeomorphisms on a torus. *Ill. J. Math.* **9** 59–65.
6. M.-E. Hamstrom: Homotopy properties of the space of homeomorphisms on P^2 and the Klein bottle. *Trans. Am. Math. Soc.* **120** (1965), 37–45.
7. M.-E. Hamstrom: Homotopy groups of the space of homeomorphisms on a 2-manifold, *Ill. J. Math.* **10** (1966), 563–573.
8. W. Heil: On P^2-irreducible 3-manifolds. *Bull. Am. Math. Soc.* **75** (1969), 772–775.
9. H. Hendriks: Applications de la theorie d'obstruction en dimension 3. *Bull. Soc. Math. France Memo.* **53** (1977) 81–196.
10. K. Johannson: *Homotopy Equivalences of 3-Manifolds with boundaries*, Springer-Verlag Lecture Notes in Mathematics, Vol. 761, Springer-Verlag, Berlin, (1979).
11. J. Kalliongis and D. McCullough: π_1-injective mappings of compact 3-manifolds. To appear in *Proc. Lond. Math. Soc.*
12. R. Kirby: Problems in low-dimensional manifold theory, in *Proceedings of Symposia in Pure Mathematics*, Volume 32, part 2, American Mathematical Society (1978) pp. 273–312.
13. R. Kramer: The twist group of an orientable cube-with-two-handles is not finitely generated, to appear.
14. E. Luft: Actions of the homeotopy group of an orientable 3-dimensional handlebody. *Math. Ann.* **234** (1978) 279–292.
15. D. McCullough: Homotopy groups of the space of self-homotopy-equivalences. *Trans. Am. Math. Soc.* **264** (1981) 151–163.
16. D. McCullough: Compact 3-manifolds with infinitely-generated groups of self-homotopy-equivalences. *Proc. Am. Math. Soc.* **91** (1984), 625–629.
17. D. McCullough and A. Miller: Homeomorphisms of 3-manifolds with compressible boundary, to appear in *Mem. Am. Math. Soc.*
18. F. Waldhausen: On irreducible 3-manifolds which are sufficiently large. *Ann. Math.* **87** (1978) 56–88.

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